Surface tension and buoyancy effects in cellular convection

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The cells observed by Bénard (1901) when a horizontal layer of fluid is heated from below were explained by Rayleigh (1916) in terms of buoyancy, and by Pearson (1958) in terms of surface tension. These rival theories are now combined. Linear perturbation techniques are used to derive a sixth-order differential equation subject to six boundary conditions. A Fourier series method has been used to obtain the eigenvalue equation for the case where the lower boundary surface is a rigid conductor and the upper free surface is subject to a general thermal condition. Numerical results are presented. It was found that the two agencies causing instability reinforce one another and are tightly coupled. Cells formed by surface tension are approximately the same size as those formed by buoyancy. Bénard's experiments are briefly discussed.

1. Introduction

The phenomena of cellular convection discovered by Bénard (1901) have attracted the attention of many writers. A fundamental theoretical paper is that of Lord Rayleigh (1916), who considered instability due to the buoyancy resulting from the expansion of a heated fluid. Later workers, including Jeffreys (1926, 1928), Low (1929), and Pellew & Southwell (1940), have extended and refined Rayleigh's analysis. The agreement with experiments involving marginal stability has been generally good. Further physical effects, such as those associated with a magnetic field, have been included by later authors. In all these treatments the agency causing instability has been buoyancy.

Pearson (1958) neglected buoyancy but offered a new explanation for the instability. He showed that if the upper surface was free then Bénard type cells could be produced by tractions arising from the variation with temperature of surface tension. He argued that in many of Bénard's experiments the cells observed must have been due to a surface-tension effect rather than buoyancy. Pearson's analysis involved a more complicated set of boundary conditions, but a simpler differential equation system, than the corresponding Rayleigh relations.

Associated with buoyancy there is a dimensionless quantity (the Rayleigh number) which takes a critical value at the onset of instability. Similarly, when Bénard cells are formed by surface tension, a dimensionless 'Marangoni number' (see §3) takes a critical value. Usually in practice both buoyancy and surface tension are operative, so it is natural to ask how the two effects are coupled.

In this paper a Fourier series method is adapted to this problem in its linear formulation. For illustration a special case is considered, namely that where the lower boundary is a rigid 'perfectly conducting' plane while the upper free surface is subject to a more general boundary condition. Small modifications only are necessary to include the effect of a vertical magnetic field. The author has performed some preliminary calculations for this case, but a magnetic field is not considered in the present paper.

2. The basic equations

As the book by Chandrasekhar (1961) is likely to become a standard reference on the Bénard problem, his notation will here be followed as far as possible. Applying a first-order perturbation technique, he indicates how the basic fluid equations of motion and heat conduction reduce to

$$\frac{\partial}{\partial t}\nabla^2 w = g\alpha\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)+\nu\nabla^4 w, \tag{2.1}$$

and

$$\partial\theta/\partial t = \beta w+\kappa\nabla^2\theta. \tag{2.2}$$

Here Cartesian axes $OXYZ$ have been taken so that the fluid is confined between the lower plane $z = 0$ and the upper plane $z = d$. The variables w and θ represent respectively the z-component of the velocity and the temperature perturbation from a uniform vertical temperature gradient. The gravitational acceleration g, the coefficient of volume expansion α, the kinematic viscosity ν, the coefficient of thermometric conductivity κ, and the adverse temperature gradient β are each assumed constant. As usual ∇^2 represents the Laplacian operator

$$\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$$

and t represents time.

In addition Pearson has assumed that the surface tension S of the liquid and Q (the rate of heat loss per unit area from the upper free surface) can be expanded to the first order in powers of θ_s (the temperature variation at the surface) in the form

$$S = S_0-\sigma_0\theta_s, \tag{2.3}$$

$$Q = Q_0+q_0\theta_s. \tag{2.4}$$

Here S_0 and Q_0 are the unperturbed values. Denoting the temperature by T we see that $-\sigma_0 = (\partial S/\partial T)$ and $q_0 = (\partial Q/\partial T)$, each partial derivative being evaluated at the unperturbed surface temperature. Thus $-\sigma_0$ represents the rate of change of surface tension with temperature, and q_0 the rate of change with temperature of the time rate of heat loss per unit area from the upper surface. For most liquids σ_0 is positive since as the temperature rises the difference between a liquid and its vapour phase decreases. The author knows of no exception.

We now consider the boundary conditions. The lower boundary will be supposed rigid and there the non-slip conditions

$$w = 0 \quad \text{and} \quad \partial w/\partial z = 0 \tag{2.5}$$

must hold. The latter condition follows from $u = v = 0$, where u and v are the x- and y-components of the velocity. Pearson showed that the appropriate conditions at the upper surface are

$$w = 0 \quad \text{and} \quad \rho\nu\frac{\partial^2 w}{\partial z^2} = \sigma_0\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right), \tag{2.6}$$

where ρ is the density of the liquid. The last relation is obtained by equating the change in surface traction (due to the temperature variations across the surface) to the shear stress experienced by the liquid at the surface. The effect of surface tension on the normal stress condition is neglected.

In his presentation Chandrasekhar has invariably assumed that the boundary surfaces are perfectly conducting, so that $\theta = 0$ on the boundaries. More generally (following Pearson) the condition on θ may be taken as

$$\theta = Y\partial\theta/\partial z, \tag{2.7}$$

where Y is a constant depending on the thermal properties of the boundary and liquid. The extreme cases $Y = 0$ and $Y^{-1} = 0$ are limiting approximations to a very good conductor (for temperature perturbations) and to a very bad one, respectively. For convenience these are referred to as the 'conducting' and 'insulating' cases. In particular, considering conservation of heat on transport across the upper surface, we have the equality

$$-k_c\partial\theta/\partial z = q_0\theta, \tag{2.8}$$

where k_c is the coefficient of heat conduction in the liquid. In general, a similar condition may apply at the lower surface, which is here taken to be conducting.

3. Normal mode analysis

We now analyse an arbitrary disturbance in terms of normal modes, supposing that the perturbations w and θ have the forms

$$w = W(z)\exp\left[i(k_x x+k_y y)+pt\right], \tag{3.1}$$

$$\theta = \Theta(z)\exp\left[i(k_x x+k_y y)+pt\right], \tag{3.2}$$

where $k = (k_x^2+k_y^2)^{\frac{1}{2}}$ is the wave-number of the disturbance and p is a constant (which can be complex). Equations (2.1) and (2.2) now become

$$p(d^2/dz^2-k^2)\,W = -g\alpha k^2\Theta+\nu(d^2/dz^2-k^2)^2\,W, \tag{3.3}$$

$$p\Theta = \beta W+\kappa(d^2/dz^2-k^2)\,\Theta. \tag{3.4}$$

These equations must be solved subject to the following boundary conditions. If the lower boundary is a rigid conductor then at $z = 0$,

$$W = 0, \quad dW/dz = 0, \quad \text{and} \quad \Theta = 0; \tag{3.5}$$

and if the upper surface is free then at $z = d$,

$$W = 0, \quad d^2W/dz^2 = -(\sigma_0 k^2/\rho\nu)\,\Theta, \quad \text{and} \quad \Theta = -(k_c/q_0)\,d\Theta/dz. \tag{3.6}$$

We shall make all quantities dimensionless by choosing d/π as the unit of length, $d^2/\pi^2\nu$ as the unit of time, $\pi\nu/d$ as the unit of velocity and $(\beta d/\pi)\,(\nu/\kappa)$ as the unit of temperature, and putting $b = kd/\pi$, $\sigma_1 = pd^2/\pi^2\nu$, $W_1 = Wd/\pi\nu$ and

$$\Theta_1 = \Theta\pi\kappa/\beta d\nu.$$

We now let x, y, z stand for co-ordinates in the new unit of length. If we denote d/dz by D, equations (3.3) and (3.4) become

$$(D^2-b^2)(D^2-b^2-\sigma_1)W_1 = (R/\pi^4)b^2\Theta_1, \tag{3.7}$$

and

$$(D^2-b^2-\mathfrak{p}\sigma_1)\Theta_1 = -W_1, \tag{3.8}$$

where $R = g\alpha\beta d^4/\kappa\nu$ is the Rayleigh number and $\mathfrak{p} = \nu/\kappa$ is the Prandtl number. In terms of the new variables and units the boundary conditions are

$$W_1 = 0, \quad DW_1 = 0, \quad \text{and} \quad \Theta_1 = 0 \quad \text{at} \quad z = 0, \tag{3.9}$$

and

$$W_1 = 0, \quad D^2W_1 = -(B/\pi^2)b^2\Theta_1, \quad \text{and} \quad D\Theta_1 = -(L/\pi)\Theta_1 \quad \text{at} \quad z = \pi, \tag{3.10}$$

where $B = \sigma_0\beta d^2/\rho\nu\kappa$ and $L = q_0 d/k_c$ are further dimensionless constants introduced by Pearson. In the chemical engineering literature B has been called the Marangoni number.

Equations (3.7) to (3.10) may be compared with Pearson's equations (15) to (19). Allowing for the changes in units and notation, the latter are equivalent to ours when we put $R = 0$.

We shall follow previous authors by setting $\sigma_1 = 0$ to obtain the equations relevant to marginal stability. Equations (3.7) and (3.8) now reduce to

$$(D^2-b^2)^2 W_1 = (R/\pi^4)b^2\Theta_1 \tag{3.11}$$

and

$$(D^2-b^2)\Theta_1 = -W_1. \tag{3.12}$$

Solutions to these equations must be found subject to the boundary conditions (3.9) and (3.10). Thus we have an eigenvalue system of sixth order. We are interested in the eigenvalue equation, which is here a relationship between the parameters b, R, B and L. We may fix the values of B and L and then minimise R as a function of the wave-number b to obtain the critical Rayleigh number for the onset of cellular convection. Alternatively, we may give R and L fixed values and obtain a critical value of the Marangoni number B by finding its minimum as b varies. By either means we can then plot curves of B versus R, for given values of L, corresponding to marginal stability.

When $R = 0$ (the case treated by Pearson) the general solutions of equations (3.11) and (3.12) may be easily written down, and the arbitrary constants involved then found by satisfying the boundary conditions. When $R \neq 0$ Pearson's method leads to very heavy algebra.

4. A Fourier series method

The method described below is an adaption of one used by Goldstein (1936, 1937) for viscous-flow instability problems and by Jeffreys & Jeffreys (1946) for the Bénard problem without surface tension.† Jeffreys & Jeffreys' boundary conditions were particularly simple, but we shall show how a more complicated case can be treated.

† This method has also been used by Jenssen (1963).

Under certain conditions the derivative of a Fourier series may be easily obtained. Following Goldstein and Jeffreys, we note that in the range $0 < x < \pi$ the derivative of the series

$$G(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{4.1}$$

is

$$G'(x) = \sum_{n=1}^{\infty} nb_n \cos nx, \tag{4.2}$$

provided that $G(0) = G(\pi) = 0$. In the same range the derivative of the series

$$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \tag{4.3}$$

is

$$F'(x) = \sum_{n=1}^{\infty} (-na_n) \sin nx, \tag{4.4}$$

without specification of $F(0)$ and $F(\pi)$. Thus differentiating the series (4.2) we obtain

$$G''(x) = \sum_{n=1}^{\infty} (-n^2 b_n) \sin nx. \tag{4.5}$$

Then

$$G'''(x) \sum_{n=1}^{\infty} (-n^3 b_n) \cos nx, \tag{4.6}$$

provided that $G''(0) = G''(\pi) = 0$, and so on. Thus we may differentiate s times the series (4.1) provided that $G(x)$ and all its even derivatives of order less than s vanish at both ends of the range.

However, even if these conditions are not satisfied by a given function, we may construct an auxiliary function which does satisfy the conditions. For example, suppose we wish to differentiate a function $f(x)$ four times. By adding a polynomial (here a cubic) we construct a function $g(x)$ satisfying

$$g(0) = g(\pi) = g''(0) = g''(\pi) = 0,$$

and expand $g(x)$ as a sine series. Thus we write

$$g(x) \equiv f(x) - \pi^{-1}[f(\pi)\, x + f(0)\,(\pi - x)] - (6\pi)^{-1}[f''(\pi)\, x(x^2 - \pi^2) - f''(0)\, x(x-\pi) \\ \times (x - 2\pi)] = \sum_{n=1}^{\infty} b_n \sin nx. \tag{4.7}$$

Then, since $g(0) = g(\pi) = 0$,

$$g'(x) \equiv f'(x) - \pi^{-1}[f(\pi) - f(0)] - (6\pi)^{-1} \\ \times [f''(\pi)\,(3x^2 - \pi^2) - f''(0)\,(3x^2 - 6\pi x + 2\pi^2)] = \sum_{n=1}^{\infty} nb_n \cos nx, \tag{4.8}$$

and

$$g''(x) \equiv f''(x) - \pi^{-1}[f''(\pi) x + f''(0)\,(\pi - x)] = \sum_{n=1}^{\infty} (-n^2 b_n) \sin nx. \tag{4.9}$$

Again, since $g''(0) = g''(\pi) = 0$,

$$g'''(x) \equiv f'''(x) - \pi^{-1}[f''(\pi) - f''(0)] = \sum_{n=1}^{\infty} (-n^3 b_n) \cos nx, \tag{4.10}$$

and

$$g^{(\mathrm{iv})}(x) \equiv f^{(\mathrm{iv})}(x) = \sum_{n=1}^{\infty} n^4 b_n \sin nx. \tag{4.11}$$

Now the expressions for $f(x), f'(x), \ldots, f^{(\mathrm{iv})}(x)$ may be used to satisfy any boundary condition involving a linear combination of these derivatives, but in order that

$f(x)$ may be made to satisfy a differential equation the polynomials must first be expanded in the usual way as Fourier series. For example, in the range $0 < x < \pi$ we get

$$x = \sum_{n=1}^{\infty} (-1)^n (2/n) \sin nx, \quad \text{and} \quad \pi - x = \sum_{n=1}^{\infty} (2/n) \sin nx,$$

so that
$$f''(x) = \sum_{n=1}^{\infty} \{(2/\pi n)[-(-1)^n f''(\pi) + f''(0)] - n^2 b_n\} \sin nx. \tag{4.12}$$

Similarly we find that

$$f(x) = \sum_{n=1}^{\infty} \{(2/\pi n)[-(-1)^n f(\pi) + f(0)] - (2/\pi n^3)[-(-1)^n f''(\pi) + f''(0)] + b_n\} \sin nx. \tag{4.13}$$

The last two formulas may be obtained more directly, but they are not always suitable for satisfying the boundary conditions if elimination of $f(0), f(\pi)$, etc., is later required.

5. Solution of the differential equations

This Fourier series method is now applied to the equations derived in § 3. These are

$$(D^2 - b^2)^2 W_1 = R_1 b^2 \Theta_1, \tag{5.1}$$

and
$$(D^2 - b^2)\Theta_1 = -W_1, \tag{5.2}$$

subject to the boundary conditions

$$W_1(0) = 0, \quad DW_1(0) = 0, \quad \Theta_1(0) = 0, \tag{5.3a, b, c}$$
$$W_1(\pi) = 0, \quad D^2W_1(\pi) = -B_1 b^2 \Theta_1(\pi), \quad D\Theta_1(\pi) = -L_1 \Theta_1(\pi), \tag{5.4a, b, c}$$

where we have written $R_1 = R/\pi^4$, $B_1 = B/\pi^2$, and $L_1 = L/\pi$.

Since we wish to differentiate W_1 four times and Θ_1 twice we write (cf. equations (4.12) and (4.13))

$$W_1 = \sum_{n=1}^{\infty} \{a_n - (2/\pi n^3)[-(-1)^n D^2W_1(\pi) + D^2W_1(0)]\} \sin nz \tag{5.5}$$

and
$$\Theta_1 = \sum_{n=1}^{\infty} \{A_n - (2/\pi n)(-1)^n \Theta_1(\pi)\} \sin nz, \tag{5.6}$$

where we have used the boundary conditions (5.3a), (5.3c) and (5.4a). Then

$$D^2W_1 = \sum_{n=1}^{\infty} \{-n^2 a_n + (2/\pi n)[-(-1)^n D^2W_1(\pi) + D^2W_1(0)]\} \sin nz, \tag{5.7}$$

$$D^4W_1 = \sum_{n=1}^{\infty} n^4 a_n \sin nz, \tag{5.8}$$

and
$$D^2\Theta_1 = \sum_{n=1}^{\infty} (-n^2 A_n) \sin nz. \tag{5.9}$$

For convenience we put $D^2W_1(0) = M$ and $D^2W_1(\pi) = N$, and then from equation (5.4b) we have $\Theta_1(\pi) = -(B_1 b^2)^{-1} N$. The remaining two boundary conditions involve odd-order derivatives. Rather than substituting directly from equations (5.5) and (5.6), it is more convenient to use alternative forms. Considering equation (4.8) we have

$$DW_1 = (6\pi)^{-1}[D^2W_1(\pi)(3z^2-\pi^2) - D^2W_1(0)(3z^2-6\pi z+2\pi^2)] + \sum_{n=1}^{\infty} na_n \cos nz, \tag{5.10}$$

and
$$D\Theta_1 = \pi^{-1}\Theta_1(\pi) + \sum_{n=1}^{\infty} nA_n \cos nz. \tag{5.11}$$

Now the boundary conditions (5.3*b*) and (5.4*c*) give

$$\sum_{n=1}^{\infty} na_n = \tfrac{1}{3}\pi M + \tfrac{1}{6}\pi N, \tag{5.12}$$

and
$$\sum_{n=1}^{\infty} (-1)^n nA_n = (L_1+\pi^{-1})(B_1b^2)^{-1}N. \tag{5.13}$$

In order to satisfy the differential equations we return to the complete Fourier series expressions (5.5) and (5.6). Substituting into equations (5.1) and (5.2) and equating coefficients of $\sin nz$ we obtain

$$(n^2+b^2)^2 a_n - R_1 b^2 A_n = \left(\frac{4b^2}{\pi n}+\frac{2b^4}{\pi n^3}\right)M + \left(\frac{2R_1}{\pi n B_1}-\frac{4b^2}{n\pi}-\frac{2b^4}{\pi n^3}\right)(-1)^n N, \tag{5.14}$$

$$a_n - (n^2+b^2)A_n = \frac{2}{\pi n^3}M + \left(\frac{2}{\pi n B_1}-\frac{2}{\pi n^3}\right)(-1)^n N. \tag{5.15}$$

Solving these two equations for a_n and A_n and substituting in equations (5.12) and (5.13) we get two homogeneous linear equations for M and N. Eliminating M and N we obtain the eigenvalue equation as the determinant of coefficients. After simplification this eigenvalue equation may be written as

$$\begin{vmatrix} \displaystyle\sum_{n=1}^{\infty} \frac{n^2(n^2+b^2)}{(n^2+b^2)^3-R_1b^2}, & \displaystyle B_1b^2\sum_{n=1}^{\infty}\frac{(-1)^n n^2(n^2+b^2)}{(n^2+b^2)^3-R_1b^2} + \sum_{n=1}^{\infty}\frac{(-1)^n R_1 b^2 n^2}{(n^2+b^2)^3-R_1b^2} \\ \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{(n^2+b^2)^3-R_1b^2}, & \displaystyle B_1b^2\sum_{n=1}^{\infty}\frac{n^2}{(n^2+b^2)^3-R_1b^2} - \frac{1+\pi L_1}{2} - \sum_{n=1}^{\infty}\frac{b^2(n^2+b^2)^2-R_1b^2}{(n^2+b^2)^3-R_1b^2} \end{vmatrix} = 0, \tag{5.16}$$

from which B_1 can be found in terms of b, L_1 and R_1 as the ratio of two determinants.

If $R_1 \neq 0$ the summation of the above series must almost certainly be done numerically. If $R_1 = 0$, which is the case considered by Pearson, each series can be summed in terms of hyperbolic functions. Thus, since $a = \pi b$,

$$\sum_{n=1}^{\infty}\frac{n^2}{(n^2+b^2)^2} = \frac{\pi^2}{4a}(\coth a - a\,\mathrm{cosech}^2 a), \tag{5.17}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{(n^2+b^2)^2} = \frac{\pi^2}{4a}(\mathrm{cosech}\, a - a\,\mathrm{cosech}\, a\coth a), \tag{5.18}$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(n^2+b^2)^3} = \frac{\pi^4}{16a^3}(\coth a + a\,\mathrm{cosech}^2 a - 2a^2\,\mathrm{cosech}^2 a\coth a), \tag{5.19}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{(n^2+b^2)^3} = \frac{\pi^4}{16a^3}(\mathrm{cosech}\, a + a\,\mathrm{cosech}\, a\coth a - a^2\,\mathrm{cosech}^3 a - a^2\,\mathrm{cosech}\, a\coth^2 a), \tag{5.20}$$

$$\sum_{n=1}^{\infty}\frac{b^2}{n^2+b^2} = \tfrac{1}{2}(a\coth a - 1). \tag{5.21}$$

Equation (5.17) then reduces (after some manipulation) to

$$B = \frac{8a(a\cosh a + L\sinh a)\,(a - \sinh a\cosh a)}{a^3\cosh a - \sinh^3 a}. \tag{5.22}$$

This is Pearson's equation (27) when misprints in the latter are corrected. (In equation (5.22) we have re-introduced $B = \pi^2 B_1$ and $L = \pi L_1$.)

For numerical calculation when $R_1 \neq 0$, it is convenient to split each series into two parts, the first being that for $R_1 = 0$. The second part is in each case more rapidly convergent. For example

$$\sum_{n=1}^{\infty} \frac{n^2(n^2+b^2)}{(n^2+b^2)^3 - R_1 b^2} = \sum_{n=1}^{\infty} \frac{n^2}{(n^2+b^2)^2} + \sum_{n=1}^{\infty} \frac{R_1 b^2 n^2}{(n^2+b^2)^2\,[(n^2+b^2)^3 - R_1 b^2]}. \tag{5.23}$$

The method described above to obtain the eigenvalue equation appears to be especially applicable to an ordinary linear differential equation involving derivatives all of even order (or all odd) with constant coefficients, subject to boundary conditions involving a linear combination of derivatives. (It was convenient here to sum certain series in terms of elementary functions, but this is not essential.) The order of the determinant obtained is then equal to the number of boundary conditions which contain at least one odd-order derivative. For example, if the boundary condition $\Theta_1(0) = 0$ were replaced by $D\Theta_1(0) = K_1\Theta_1(0)$ (thus generalizing the thermal boundary condition at the lower surface), then a 3×3 determinant would be obtained.

6. Numerical results

An IBM 1620 digital computer was programmed to calculate B_1 for various values of b, L_1 and R_1. The minimum of B_1 with respect to b (for given values of L_1 and R_1) was obtained by interpolation. For presentation, the results have been expressed in terms of the more familiar parameters a, R, B and L.

The (R, B)-locus corresponding to marginal stability is plotted in figure 1, for each of the limiting cases $L = 0$ and $L = \infty$. (The values plotted are the critical values at the appropriate critical wave-numbers.) For intermediate values of L the loci lie between these two curves, the curvature increasing monotonically with increase in L. The region where $R > 0$ but $B < 0$ is applicable to a layer of liquid adhering to a ceiling which is cooler than the air below, or to a liquid whose coefficient of volume expansion is negative (for example, water at a temperature between 0 and 4 °C) cooled from below. That region where $R < 0$ but $B > 0$ is applicable to the opposite cases of a hotter ceiling, or a contracting liquid heated from below.

Figure 2 illustrates the variation of the corresponding dimensionless wave-numbers at marginal stability. These give the sizes of the convection cells which are formed.

In table 1 are presented the numerical values of B_c and a_B (the critical Marangoni number and wave-number when $R = 0$) and of R_c and a_R (the critical Rayleigh number and wave-number when $B = 0$), for various values of L. When L becomes large R_c tends to a finite limit, while B_c becomes asymptotically proportional to L. The wave-numbers remain finite.

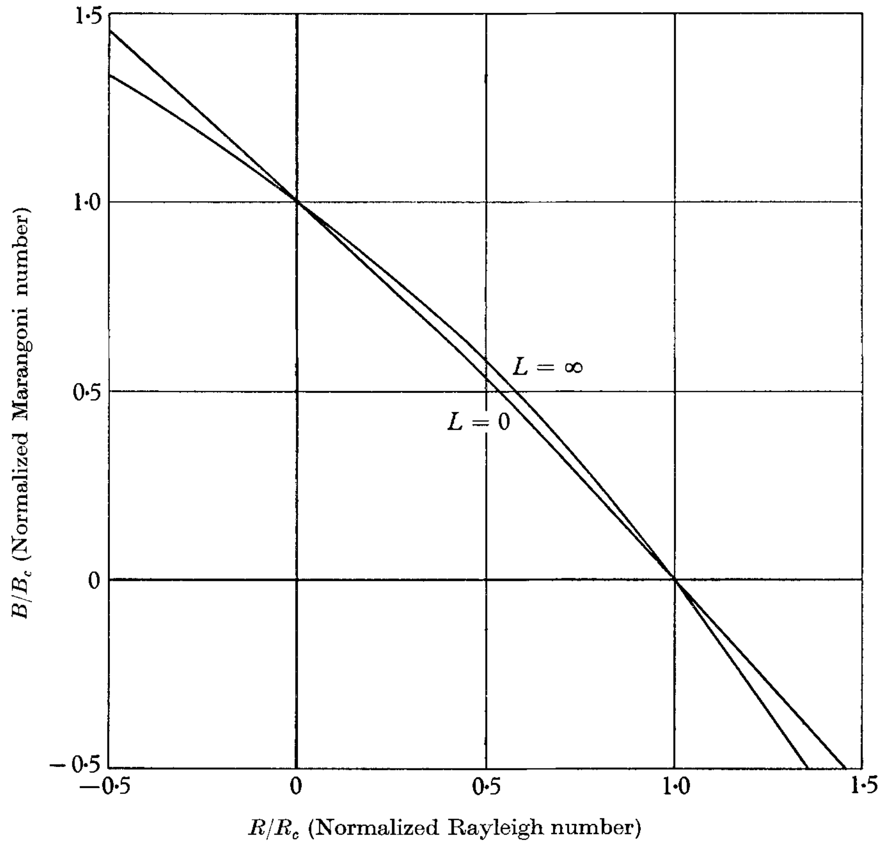

FIGURE 1. Stability diagram. Plot of Marangoni and Rayleigh numbers for marginal stability (normalized to give unit intercepts on the co-ordinate axes). $L = 0$ and $L = \infty$ refer to 'insulating' and 'conducting' free surfaces. Points below a curve represent stable states.

L	B_c	a_B	R_c	a_R
0	79·607	1·993	669·00	2·086
0·01	79·991	1·997	670·38	2·089
0·1	83·427	2·028	682·36	2·117
0·2	87·195	2·060	694·78	2·144
0·5	98·256	2·142	727·42	2·212
1	116·127	2·246	770·57	2·293
2	150·679	2·386	831·27	2·393
5	250·598	2·598	925·51	2·519
10	413·440	2·743	989·49	2·589
20	736·00	2·852	1036·30	2·632
50	1699·62	2·941	1072·19	2·661
100	3303·83	2·976	1085·90	2·672
1000	32170·1	3·010	1099·12	2·681
10^{10}	$32{\cdot}0730 \times 10^{10}$	3·014	1100·65	2·682

TABLE 1. Critical values of Marangoni number and Rayleigh number, and the corresponding wave-numbers for various values of L

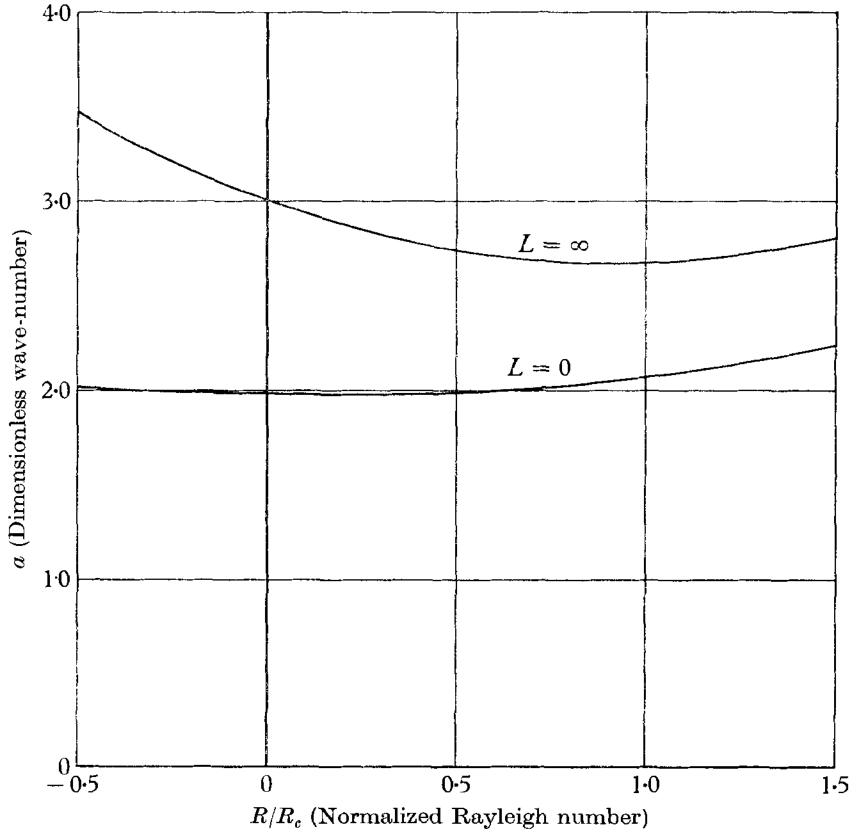

FIGURE 2. Wave-numbers corresponding to marginal stability, plotted against normalized Rayleigh number. (R_c corresponds to $B = 0$.)

7. Discussion

The values of B_c and a_B when $L = 0$ confirm those calculated by Pearson, while those of R_c and a_R when L is very large (taken here as 10^{10}) agree precisely with those calculated by Reid & Harris (1958) and quoted in Chandrasekhar's book.

From figure 1 we see that since the critical Marangoni number decreases with increase of the Rayleigh number, the two agencies causing instability (buoyancy and the variation with temperature of surface tension) reinforce each other. The small curvature, concave downwards, shows that the coupling between the two agencies is tight (in the sense that a small change in either B or R results in a change of the same order in the other) but not perfect. The following argument shows that if there were maximum reinforcement of the two agencies, one would expect the locus to be the straight line $R/R_c + B/B_c = 1$. Since R is proportional to β, and thus to the temperature difference between the upper and lower surfaces, the energy potentially available from the buoyancy effect is proportional to R. Thus for a fixed cellular size and streamline pattern, R/R_c is the ratio of the energy available from buoyancy to that required to balance the viscous dissipation at the onset of convection. Similarly on our linear theory the energy potentially available from surface tension tractions is proportional to β and thus to B. Thus B/B_c is the ratio of the energy available from surface tension to the viscous dissipation at the onset of convection. Since at marginal stability there is a balance between the kinetic energy dissipated by viscosity and the energy supplied by the buoyancy and surface tension forces, the conclusion follows.

Because of the difference in nature of the agencies, the tightness of the coupling is surprising. It is presumably a consequence of the small difference between the sizes of the cells produced by the agencies acting separately. The coupling is particularly tight when the free surface is 'insulating' ($L = 0$). For this case the values of a_B and a_R are 1·993 and 2·086, respectively. When the free surface is 'conducting' (L very large) the coupling is less tight. Now the difference in cell sizes is greater, the values for a_B and a_R being 3·014 and 2·682. This is further illustrated by figure 2.

Increase in L from 0 to ∞ means a change in the thermal boundary condition at the free surfaces from $D\Theta = 0$ to $\Theta = 0$. Since R appears in the differential equation system as an eigenvalue, general theory predicts that the critical Rayleigh number for a fixed value of B should be an increasing function of L. At the same time the critical wave-number increases, so that the size of the convection cells decreases. These trends might be forecast on physical grounds. With an insulated boundary it is easier for temperature perturbations to be set up, and a smaller vertical thermal gradient is required. Less energy must then be dissipated by vorticity to balance the reduced release of internal energy by buoyancy. Both dissipation and energy release are greater for a fine reticulation into cells. Larger cells (or smaller wave-numbers) are therefore associated with the insulating case of $L = 0$.

When the free surface is a good conductor any temperature variations across the surface decay rapidly and surface tension tractions are therefore small. Thus when L is large the critical Marangoni number must also be large. We find that asymptotically $B_c = 32{\cdot}073L$. Again, as L increases the corresponding wave-number increases, so that the size of the convection cells decreases.

The values of R_c and a_R for the case $L = 0$ are particularly interesting. These are the values of the Rayleigh number and the wave-number at the onset of instability caused by changes of density only, when the lower boundary is a 'rigid conductor' and the upper is a 'free insulating' surface. Jeffreys (1926) treated this case, but with incorrect boundary conditions, and obtained the values 571 and 3·5, respectively, in contrast to the values 669·00 and 2·086 obtained here. In a later paper Jeffreys (1928) gave the correct boundary conditions but did not repeat his calculation, and his incorrect values were the best available to Pearson.

Pearson pointed out that Bénard's experimental results led to values of a between 2·1 and 2·5, which were in better agreement with Pearson's value for a_B, namely 2·0, than with the Jeffreys value of 3·5 for a_R, suggesting that Bénard's cells were caused by surface tension alone. This argument is no longer justified, since the present results indicate little difference between a_B and a_R for a given small value of L.

However, Pearson's alternative argument still holds. This was based on the evidence that in at least some of Bénard's experiments the critical Rayleigh number R_c was not exceeded but the critical Marangoni number B_c was almost certainly exceeded. Further confirmation is provided by the experiments described by Block (1956).

It is thus probable that the cells observed by Bénard were caused mainly by surface tension. Since B contains a factor d^2 while R depends on d^4 one might expect surface tension to become more important for thin layers of liquid. (Buoyancy must, of course, be the sole agency responsible when there is no free surface.) A repetition of the original experiments in order to obtain more precise quantitative results would be welcomed.

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